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ON THE EQUATIONS OF AN EIGHTH-ORDER THEORY FOR NONHOMOGENEOUS TRANSVERSELY ISOTROPIC PLATES

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Abstract—We obtain the equations of a two-dimensional eighth-order theory for the transverse bending of transversely isotropic plates, through use of a variational equation for displacements and *transverse* stresses. We show that, for sufficiently thin plates, the solution involves an equation of the fourth order for an interior solution contribution, and *two* equations of the second order for edge zone solution contributions. One of these accounts for transverse shear stress effects, in nearly the same manner as this effect occurs in the sixth-order theory, and the other accounts for transverse normal stress effects which are neglected in the sixth-order theory.

INTRODUCTION

A recent concern with an eighth-order theory for homogeneous isotropic plates based on displacement approximations $u_x = z\phi_x$, $u_y = z\phi_y$, $u_z = w+z^2v$ by Li and Babuska (1992) has suggested the following alternate formulation of an eighth-order theory based on approximations which are in part of the displacement type and in part of the stress type. This in expectation that thereby the results should approximate more closely the consequences of a three-dimensional analysis. The developments which follow are based on a reduction of the equations of a twelfth-order theory which has previously been established through use of a variational equation for displacement and *transverse* stresses (Reissner, 1991).

Our main interest in what follows is to obtain transformations of the equilibrium and stress displacement equations of the eighth-order theory which show the separation between interior and edge zone solution contributions as well as the distinction between a transverse shear stress boundary layer and a transverse normal stress boundary layer with the latter being a consequence of the step from sixth-order theory (Reissner, 1944) to the present eighth-order theory.

A specialization of the equations in this paper to the case of homogeneous plates shows, in particular, the dependence of the widths of the two boundary layers on the ratio of the midplane Young's modulus E to the transverse shear modulus G, and on the ratio of G to the transverse Young's modulus E_z, respectively.

As an example of application we consider the stress boundary value problem for a semi-infinite plate, in analogy to the corresponding simpler sixth-order analysis.

In an Appendix, we state the associated two-dimensional variational equation which would have to be used in connection with finite element implementations of the present system of plate equations.

THE THREE-DIMENSIONAL PROBLEM AND THE EQUATIONS OF AN EIGHTH-ORDER TWO-DIMENSIONAL THEORY

The results which follow are based on the conventional equilibrium equations for stress in conjunction with constitutive equations of the form

$$u_{x,x} = \frac{\sigma_x - \nu\sigma_y}{E} - \frac{\nu_z\sigma_z}{\sqrt{EE_z}}, \quad u_{y,y} = \frac{\sigma_y - \nu\sigma_x}{E} - \frac{\nu_z\sigma_z}{\sqrt{EE_z}}, \tag{1}$$

$$u_{x,y} + u_{y,x} = 2(1+\nu)\frac{\tau_{xy}}{E}, \quad u_{z,z} = \frac{\sigma_z}{E_z} - \nu_z\frac{\sigma_x + \sigma_y}{\sqrt{EE_z}}, \tag{2}$$

$$u_{x,z} + u_{z,x} = \frac{\tau_{xz}}{G}, \quad u_{y,z} + u_{z,y} = \frac{\tau_{yz}}{G}, \tag{3}$$

with E, ν, E_z, ν_z and G as given even functions of z, and with surface conditions $\sigma_z = \pm 1/2q(x,y)$, $\tau_{xz} = \tau_{yz} = 0$ for $z = \pm c$.

To reduce this three-dimensional problem to a two-dimensional eighth-order problem we use the variational equation for displacements and transverse stresses in Reissner (1985), in conjunction with midplane parallel displacement approximations:

$$u_x = z\phi_x(x,y), \quad u_y = z\phi_y(x,y) \tag{4}$$

and midplane perpendicular stress approximations:

$$(\tau_{xz}, \tau_{yz}) = \frac{(Q_x, Q_y)}{4c/3}\left(1 - \frac{z^2}{c^2}\right) + \frac{(S_x, S_y)}{8c}\left(1 - 6\frac{z^2}{c^2} + 5\frac{z^4}{c^4}\right), \tag{5}$$

$$\sigma_z = \frac{3q}{4}\left(\frac{z}{c} - \frac{1}{3}\frac{z^3}{c^3}\right) + \frac{T}{8}\left(\frac{z}{c} - 2\frac{z^3}{c^3} + \frac{z^5}{c^5}\right). \tag{6}$$

The stress approximations (5) and (6) satisfy the boundary conditions for $z = \pm c$, as well as the transverse equilibrium equation $\tau_{xz,x} + \tau_{yz,y} + \sigma_{z,z} = 0$.

The function of z in the S-terms in (5) is the lowest even degree polynomial with the property that these terms do not make a contribution to the transverse shear stress *resultants*.

The introduction of (4)–(6) into the indicated variational equation leads to the following system of two-dimensional plate equations:

$$M_{xx,x} + M_{xy,y} = Q_x, \quad M_{xy,x} + M_{yy,y} = Q_y, \tag{7}$$

$$Q_{x,x} + Q_{y,y} + q = 0, \quad S_{x,x} + S_{y,y} + T = 0, \tag{8}$$

$$M_{xx} = D\phi_{x,x} + D_\nu\phi_{y,y} + A_T T + A_q q, \quad M_{yy} = \cdots, \tag{9}$$

$$M_{xy} = \tfrac{1}{2}(D - D_\nu)(\phi_{x,y} + \phi_{y,x}), \tag{10}$$

$$\phi_x + w_{,x} = B_Q Q_x + B_R S_x, \quad \phi_y + w_{,y} = \cdots, \tag{11}$$

$$v_{,x} = B_R Q_x + B_S S_x, \quad v_{,y} = \cdots, \tag{12}$$

$$v = A_T(\phi_{x,x} + \phi_{y,y}) - C_T T - C_q q. \tag{13}$$

In the derivation of the system (7)–(13) there is no reference to the z-dependence of the midplane parallel components of stress and of the midplane perpendicular component of displacement. Equations (9) and (10) serve as defining relations for the stress couples $\int_{-c}^{c} (\sigma_x, \sigma_y, \tau_{xy})z\,dz$, and u_z enters in the results only by way of the two weighted averages

$$w = \frac{3}{4c}\int_{-c}^{c}\left(1-\frac{z^2}{c^2}\right)u_z\,\mathrm{d}z, \quad v = \frac{1}{8c}\int_{-c}^{c}\left(1-6\frac{z^2}{c^2}+5\frac{z^4}{c^4}\right)u_z\,\mathrm{d}z. \tag{14}$$

The constitutive coefficients in (9)–(13) are, in accordance with the results in Reissner (1991), and with some suitable change of notation

$$D = \int_{-c}^{c}\frac{E}{1-\nu^2}z^2\,\mathrm{d}z, \quad D_\nu = \int_{-c}^{c}\frac{\nu E}{1-\nu^2}z^2\,\mathrm{d}z, \tag{15}$$

$$A_T = \frac{1}{8}\int_{-c}^{c}\frac{\nu_z}{1-\nu}\sqrt{\frac{E}{E_z}}\left(\frac{z}{c}-2\frac{z^3}{c^3}+\frac{z^5}{c^5}\right)z\,\mathrm{d}z, \tag{16a}$$

$$A_q = \frac{3}{4}\int_{-c}^{c}\frac{\nu_z}{1-\nu}\sqrt{\frac{E}{E_z}}\left(\frac{z}{c}-\frac{1}{3}\frac{z^3}{c^3}\right)z\,\mathrm{d}z, \tag{16b}$$

$$B_Q = \frac{9}{16c^2}\int_{-c}^{c}\left(1-\frac{z^2}{c^2}\right)^2\frac{\mathrm{d}z}{G}, \tag{17a}$$

$$B_R = \frac{3}{32c^2}\int_{-c}^{c}\left(1-\frac{z^2}{c^2}\right)\left(1-6\frac{z^2}{c^2}+5\frac{z^4}{c^4}\right)\frac{\mathrm{d}z}{G}, \tag{17b}$$

$$B_S = \frac{1}{64c^2}\int_{-c}^{c}\left(1-6\frac{z^2}{c^2}+5\frac{z^4}{c^4}\right)^2\frac{\mathrm{d}z}{G}, \tag{17c}$$

$$C_T = \frac{1}{64}\int_{-c}^{c}\left(1-\frac{2\nu_z^2}{1-\nu}\right)\left(\frac{z}{c}-2\frac{z^3}{c^3}+\frac{z^5}{c^5}\right)^2\frac{\mathrm{d}z}{E_z}, \tag{18a}$$

$$C_q = \frac{3}{32}\int_{-c}^{c}\left(1-\frac{2\nu_z^2}{1-\nu}\right)\left(\frac{z}{c}-\frac{1}{3}\frac{z^3}{c^3}\right)\left(\frac{z}{c}-2\frac{z^3}{c^3}+\frac{z^5}{c^5}\right)\frac{\mathrm{d}z}{E_z}. \tag{18b}$$

The eighth-order system (7)–(13) reduces to a version of sixth-order shear deformable plate theory upon setting $S_x = S_y = T = 0$, and $A_T = B_R = B_S = C_T = C_q = 0$.

Among possible boundary conditions for the above problem may be noted, in particular, the system of four stress conditions $M_{nn} = \bar{M}_{nn}$, $M_{ns} = \bar{M}_{ns}$, $Q_n = \bar{Q}_n$, $S_n = \bar{S}_n$ and the system of four displacement conditions $\phi_n = \bar{\phi}_n$, $\phi_s = \bar{\phi}_s$, $w = \bar{w}$, $v = \bar{v}$.

A REDUCTION OF THE EIGHTH-ORDER SYSTEM TO TWO SECOND ORDER AND ONE FOURTH ORDER DIFFERENTIAL EQUATIONS

The first step in this reduction involves the introduction of (9) and (10) into (7). The result can be written in the form

$$Q_x = D\Phi_{,x}+\tfrac{1}{2}(D-D_\nu)\Psi_{,y}+A_T T_{,x}+A_q q_{,x}, \tag{19a}$$

$$Q_y = D\Phi_{,y}-\tfrac{1}{2}(D-D_\nu)\Psi_{,x}+A_T T_{,y}+A_q q_{,y}, \tag{19b}$$

where

$$\Phi = \phi_{x,x}+\phi_{y,y}, \quad \Psi = \phi_{x,y}-\phi_{y,x}. \tag{20}$$

The introduction of (19a, b) into the first relation in (8) gives, as an intermediate result

$$D\nabla^2\Phi + A_T\nabla^2 T = -q - A_q\nabla^2 q. \tag{21}$$

As a second intermediate result we deduce, by means of an introduction of (13) into a suitably differentiated version of (12), in conjunction with (8)

$$A_T\nabla^2\Phi - C_T\nabla^2 T = -B_S T - B_R q + C_q\nabla^2 q. \tag{22}$$

Next, the elimination of w and v in (11) and (12) by cross differentiation and the observation of (19) gives as a differential equation for Ψ

$$\left(B_Q - \frac{B_R^2}{B_S}\right)\frac{D-D_v}{2}\nabla^2\Psi = \Psi. \tag{23}$$

The three equations (21)–(23) for Φ, T and Ψ are complemented by a fourth equation involving w, upon deducing from (11), in conjunction with (8), that

$$\nabla^2 w = -\Phi - B_R T - B_Q q. \tag{24}$$

Remarkably, the second order equation (23) for Ψ can be complemented by a second order equation involving only T and q. This equation is obtained upon considering (21) and (22) as two simultaneous linear equations for $\nabla^2 T$ and $\nabla^2\Phi$, in the form

$$\left(\frac{C_T}{B_S} + \frac{A_T^2}{DB_S}\right)\nabla^2 T = T + \left(\frac{B_R}{B_S} - \frac{A_T}{DB_S}\right)q - \left(\frac{C_q}{B_S} + \frac{A_T A_q}{DB_S}\right)\nabla^2 q. \tag{25}$$

The two uncoupled second-order equations for Ψ and T are complemented by one fourth-order equation for w, which follows upon introduction of (24) into (21):

$$D\nabla^4 w = q + (A_q - DB_Q)\nabla^2 q + (A_T - DB_R)\nabla^2 T. \tag{26}$$

With (23), (25) and (26) it remains to express v and the various ϕ, Q, S and M in terms of Ψ, T and w.

An introduction of (24) into (19a, b) gives as expressions for Q_x and Q_y:

$$Q_x = -[D\nabla^2 w + (DB_Q - A_q)q + (DB_R - A_T)T]_{,x} + \tfrac{1}{2}(D - D_v)\Psi_{,y}, \tag{27a}$$

$$Q_y = -[D\nabla^2 w + (DB_Q - A_q)q + (DB_R - A_T)T]_{,y} - \tfrac{1}{2}(D - D_v)\Psi_{,x}. \tag{27b}$$

Equations (13), (20) and (24) give as expression for v:

$$v = -A_T\nabla^2 w - (C_q + A_T B_Q)q - (C_T + A_T B_R)T. \tag{28}$$

With this we have S_x and S_y in terms of w, Ψ, T and q upon rewriting (12) in the form:

$$B_S S_x = v_{,x} - B_R Q_x, \quad B_S S_y = v_{,y} - B_R Q_y. \tag{29}$$

The introduction of (29) into (11) then gives as expressions for ϕ_x and ϕ_y:

$$\phi_x = -w_{,x} + \left(B_Q - \frac{B_R^2}{B_S}\right)Q_x + \frac{B_R}{B_S}v_{,x}, \tag{30a}$$

$$\phi_y = -w_{,y} + \left(B_Q - \frac{B_R^2}{B_S}\right)Q_y + \frac{B_R}{B_S}v_{,y}. \tag{30b}$$

Finally, it is apparent that the introduction of (30a, b) into (9) and (10), in conjunction with (27a, b) and (28), will result in expressions for M_{xx}, M_{yy} and M_{xy} in terms of w, Ψ, T and q.

CONSTITUTIVE COEFFICIENTS FOR TRANSVERSELY HOMOGENEOUS PLATES

In order to clarify the nature of the foregoing we list below the values of the constitutive coefficients in (15)–(18) for the case of z-independent E, ν, ν_z, E_z and G, as well as the values of the coefficients in (23) and (25) which are significant for our conclusions:

$$(D, D_\nu) = \tfrac{2}{3}(1, \nu)\frac{Ec^3}{1-\nu^2}; \quad (A_q, A_T) = \left(\frac{2}{5}, \frac{2}{105}\right)\frac{\nu_z c^2}{1-\nu}\sqrt{\frac{E}{E_z}}, \tag{31}$$

$$(B_Q, B_R, B_S) = \left(\frac{3}{5}, \frac{1}{35}, \frac{4}{315}\right)\frac{1}{Gc}, \tag{32}$$

$$(C_q, C_T) = \left(\frac{79}{1260}, \frac{4}{3465}\right)\left(1 - \frac{2\nu_z^2}{1-\nu}\right)\frac{c}{E_z}. \tag{33}$$

From (31) and (32) follows for the coefficient in (23):

$$\left(B_Q - \frac{B_R^2}{B_S}\right)\frac{D - D_\nu}{2} = \frac{1}{5}\left(1 - \frac{3}{28}\right)\frac{Ec^2}{(1+\nu)G} \equiv c_\tau^2. \tag{34}$$

From (31)–(33) follows for the first coefficient in (25):

$$\frac{C_T}{B_S} + \frac{A_T^2}{DB_S} = \frac{1}{11}\left(1 - \frac{107}{70}\,\frac{1 - 33\nu/107}{1-\nu}\,\nu_z^2\right)\frac{Gc^2}{E_z} \equiv c_\sigma^2. \tag{35}$$

We conclude from the form of (34) and (35) that Ψ, as well as the solution of the homogeneous equation for T are boundary layer portions of the solution of the given eighth-order problem for cases in which c_τ and c_σ are small compared to the representative linear dimension in the plane of the plate. The boundary layer corresponding to (34) represents the effect of transverse shear deformability, and the width of the layer is nearly the same as the corresponding width $\sqrt{E/5(1+\nu)G}c$ associated with the theory of the sixth order.

The boundary layer corresponding to (35) is a consequence of the step from sixth order to eighth order. The width of both layers is of the same order as the thickness $2c$ of the plate, with significant modifications which represent the consequences of the difference between isotropy and transvere isotropy.

With Ψ being entirely a boundary layer solution contribution, we have that T involves both a boundary layer contribution T^e and interior solution contribution T^i, with the latter being absent if there are no surface loads q.

An introduction of T into (26) shows next that in the eighth-order theory the deflection function w will have an edge zone solution contribution w^e in addition to the expected interior solution contribution w^i, corresponding to the effect of q and T in (26). The solution part w^e follows upon introducing T^e in terms of $\nabla^2 T^e$ in accordance with (25) into (26), in the form

$$\begin{aligned} w^e &= \left(\frac{A_T}{D} - B_R\right)\left(\frac{C_T}{B_S} + \frac{A_T^2}{DB_S}\right)T^e \\ &= \frac{-1}{385}\frac{c}{E_z}\left(1 - \nu_z G\frac{1+\nu}{\sqrt{EE_z}}\right)\left(1 - \frac{107 - 33\nu}{70 - 70\nu}\nu_z^2\right)T^e. \end{aligned} \tag{36}$$

As far as the solution part w^{i} is concerned we observe that (26) implies that, except for terms of relative order c^2/a^2 where a is a characteristic length for significant changes of q, w^{i} again comes out to be the solution of Kirchhoff's equation $D\nabla^4 w = q$.

ON THE DETERMINATION OF FIRST-ORDER TRANSVERSE STRESS CORRECTIONS

Knowing the existence of *first*-order transverse shear stress corrections for the fourth-order interior problem as established by Goldenveizer through a study of the three-dimensional problem and, according to Reissner (1985), as implied by the two-dimensional sixth-order problem, it is of interest also to consider this effect in the context of the present eighth-order theory.

With $q = 0$, we consider a semi-infinite plate $-\infty \leqslant x \leqslant 0$ with loading conditions

$$Q_x = \bar{Q}_x, \quad S_x = \bar{S}_x, \quad M_{xx} = \bar{M}_{xx}, \quad M_{xy} = \bar{M}_{xy}, \tag{37}$$

for $x = 0$. The load system is to be self equilibrating, so that $Q_x = S_x = M_{xx} = M_{xy} = 0$ for $x = -\infty$. Furthermore, we stipulate a characteristic length a for edgewise rates of change of the right side terms in (37) such that $a \gg c$.

With $w = w^{\mathrm{i}} + w^{\mathrm{e}}$ we have from (27a) and (26) after a remarkable cancellation of T and w^{e} as expression for Q_x in (37), except for terms of relative order c^2/a^2:

$$Q_x = -D\nabla^2 w^{\mathrm{i}}_{,x} + \tfrac{1}{2}(D - D_\nu)\Psi_{,y}. \tag{38}$$

With this we have from (29) and (28) as expression for S_x in (37)

$$S_x = \frac{DB_R - A_T}{B_S}\nabla^2 w^{\mathrm{i}}_{,x} - \frac{B_R}{B_S}\frac{D-D_\nu}{2}\Psi_{,y} - \frac{DC_T + A_T^2}{DB_S}T_{,x}. \tag{39}$$

To obtain the corresponding formulae for M_{xx} and M_{xy} we first obtain ϕ_x and ϕ_y. The introduction of Q_x and S_x from (38) and (39) with the corresponding formulae for Q_y and S_y, and the introduction of v from (29) into (30a, b) gives, except for term of relative order c^2/a^2:

$$\phi_x = -w^{\mathrm{i}}_{,x} + \frac{B_Q B_S - B_R^2}{B_S}\frac{D-D_\nu}{2}\Psi_{,y} - \frac{DC_T + A_T^2}{DB_S}\frac{A_T}{D}T_{,x} \tag{40}$$

with ϕ_y correspondingly.

We next observe that as a consequence of our stipulation concerning edgewise rates of changes and of the form of the differential equations for Ψ and T these may here be used in the abbreviated form

$$\frac{B_Q B_S - B_R^2}{B_S}\frac{D-D_\nu}{2}\Psi_{,xx} = \Psi, \quad \frac{DC_T + A_T^2}{DB_S}T_{,xx} = T. \tag{41}$$

From (41) it follows that, except for terms of relative order c^2/a^2,

$$\Psi_{,x} = \sqrt{\frac{B_S}{B_Q B_S - B_R^2}\frac{2}{D-D_\nu}}\,\Psi, \quad T_{,x} = \sqrt{\frac{DB_S}{DC_T + A_T^2}}\,T. \tag{42}$$

The introduction of (42) and (40) into (9) then gives, after a remarkable cancellation of all T-terms, as expression for M_{xx}

$$M_{xx} = M^{\mathrm{i}}_{xx} + (D-D_v)\sqrt{\frac{B_Q B_S - B_R^2}{B_S}\frac{D-D_v}{2}}\,\Psi_{,y}, \tag{43}$$

where $M^{\mathrm{i}}_{xx} = -Dw^{\mathrm{i}}_{,xx} - D_v w^{\mathrm{i}}_{,yy}$.

Correspondingly, from (10)

$$M_{xy} = M^{\mathrm{i}}_{xy} - \frac{D-D_v}{2}\Psi - A_T\frac{D-D_v}{D}\sqrt{\frac{DC_T + A_T^2}{DB_S}}\,T_{,y}, \tag{44}$$

where $M^{\mathrm{i}}_{xy} = -(D-D_v)w^{\mathrm{i}}_{,xy}$.

The introduction of (44), (43), (39) and (38) into (37) leaves a system of four coupled boundary conditions for the three functions w^{i}, Ψ and T. From this we now deduce a system of two conditions for w^{i} by itself, which hold except for terms of relative order c^2/a^2, while including terms of relative order c/a.

A comparison of the influence of the various terms in the four relevant expressions leads to the initial conclusion that the $T_{,y}$-term in (44) is of relative order c^2/a^2. The omission of this term makes possible a contraction of the two conditions for Q_x and M_{xy} in (37) into the contracted Kirchhoff condition

$$Q^{\mathrm{i}}_x - \bar{Q}_x + M^{\mathrm{i}}_{xy,y} - \bar{M}_{xy,y} = 0. \tag{45}$$

A second condition for w^{i} follows upon introducing (47), without the $T_{,y}$-term, into the M_{xx} condition in (40), in the form

$$M^{\mathrm{i}}_{xx} - \bar{M}_{xx} + \sqrt{\frac{B_Q B_S - B_R^2}{B_S}\frac{D-D_v}{2}}\,2(M^{\mathrm{i}}_{xy,y} - \bar{M}_{xy,y}) = 0, \tag{46}$$

with the M_{xy}-term in (49) representing a modification of relative order c/a of the corresponding Kirchhoff condition.

Having determined w^{i} on the basis of (48) and (49) we subsequently determine Ψ and T through the conditions

$$\Psi = \frac{2}{D-D_v}(M^{\mathrm{i}}_{xy} - \bar{M}_{xy}), \tag{47}$$

$$T = +\sqrt{\frac{DB_S}{DC_T + A_T^2}}\left[(S^{\mathrm{i}}_x - \bar{S}_x) - \frac{B_R}{B_S}(M^{\mathrm{i}}_{xy,y} - \bar{M}_{xy,y})\right], \tag{48}$$

for $x = 0$.

Two salient consequences of the foregoing may be stated as follows:

(1) Conditions (45) and (46) for the interior solution contribution in the eighth-order formulation are the same as the corresponding conditions in the sixth-order formulation, except for a (numerically small) change of the term B_Q into a term $B_Q - B_R^2/B_S$. There is, to this order of magnitude, no influence of transverse normal stress.

(2) The effect of the S_x stipulation in (37) shows, on the basis of (39), that the nature of the transverse edge shear distribution in accordance with (5) plays a significant role not only in the edge zone but also in the interior. Physically, the magnitude of this effect involves a transverse normal stress influence through the occurrence of the parameter $\nu_z G/\sqrt{EE_z}$.

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APPENDIX

In connection with possible finite element implementations we state below the variational equation corresponding to the differential equations (7)–(13), including a fairly general system of edge conditions. With all differential equations and edge conditions as Euler equations we have $\delta I = 0$, with

$$\begin{aligned} I = \iint &[\tfrac{1}{2}D(\phi_{x,x}^2 + \phi_{y,y}^2) + D_v \phi_{x,x}\phi_{y,y} + \tfrac{1}{2}(D - D_v)(\phi_{x,y} + \phi_{y,x})^2 \\ &+ (\phi_{x,x} + \phi_{y,y})(A_q q + A_T T) - C_q q T - \tfrac{1}{2}C_T T^2 \\ &- \tfrac{1}{2}B_Q(Q_x^2 + Q_y^2) - B_R(Q_x S_x + Q_y S_y) - \tfrac{1}{2}B_S(S_x^2 + S_y^2) \\ &+ (\phi_x + w_{,x})Q_x + \cdots + v_{,x}S_x + \cdots - qw - Tv]\,\mathrm{d}x\,\mathrm{d}y \\ - \int &[\bar{M}_{nn}\phi_n\,\mathrm{d}s_{\phi n}^s + \bar{M}_{ns}\phi_s\,\mathrm{d}s_{\phi s}^s + \bar{Q}_n w\,\mathrm{d}s_w^s + \bar{S}_n v\,\mathrm{d}s_v^s \\ &+ (\sigma_n - \bar{\phi}_n)M_{nn}\,\mathrm{d}s_{\phi n}^d + (\phi_s - \bar{\phi}_s)M_{ns}\,\mathrm{d}s_{\phi s}^d + (w - \bar{w})Q_n\,\mathrm{d}s_w^d + (v - \bar{v})S_n\,\mathrm{d}s_v^d]. \end{aligned} \quad (49)$$

In this we have $s_{\phi n}^s + s_{\phi n}^d = s$, etc. with s as arc length along the edge of the plate, and with the superscripts referring to stress or displacement conditions.